

4. Use the definition of a Laplace transform to find  $\mathcal{L}\{t e^{2t}\}$

$$\begin{aligned} \int_0^\infty t e^{2t} e^{-st} dt &= \int_0^\infty t e^{(2-s)t} dt \\ &= \left[ \frac{1}{2-s} t e^{(2-s)t} - \frac{1}{(2-s)^2} e^{(2-s)t} \right]_0^\infty \\ &= \frac{1}{(2-s)^2} \end{aligned}$$

$$\begin{array}{ll} t & e^{(2-s)t} \\ 1 & \frac{1}{2-s} e^{(2-s)t} \\ 0 & \frac{1}{(2-s)^2} e^{(2-s)t} \end{array}$$

2. Find the Laplace transforms of the following functions.

4. A.  $f(t) = 2t^5 + \sin(3t)$

$$F(s) = \frac{240}{s^6} + \frac{3}{s^2+9}$$

5. B.  $f(t) = 2t - 3 + e^{-8t}$

$$F(s) = \frac{2}{s^2} - \frac{3}{s} + \frac{1}{s+8}$$

4. C.  $f(t) = e^{3t} \cos(4t)$

$$F(s) = \frac{s-3}{(s-3)^2 + 16}$$

5. D.  $f(t) = 3 - 3U(t-2) + 4t^2 U(t-2)$

$$\begin{aligned} F(s) &= \frac{3}{s} - \frac{3}{s} e^{-2s} + 4 \int_2^\infty (t+2)^2 e^{-2s} dt \\ &= \frac{3}{s} - \frac{3}{s} e^{-2s} + \left[ \frac{8}{s^3} + \frac{16}{s^2} + \frac{16}{s} \right] e^{-2s} \end{aligned}$$

6. E.  $f(t) = t \cosh(7t)$

$$F(s) = (-1) \frac{d}{ds} \left[ \frac{s}{s^2-49} \right] = - \left[ \frac{(s^2-49) - s(2s)}{(s^2-49)^2} \right] = \frac{s^2+49}{(s^2-49)^2}$$

7. 3. Find the inverse Laplace transforms of the following.

A.  $\mathcal{L}^{-1} \left\{ \frac{6}{s} + \frac{1}{s^4} - \frac{3}{s+6} + \frac{1}{(s-7)^5} \right\} = 6 + \frac{1}{6} t^3 - 3e^{-6t} + \frac{1}{24} t^4 e^{7t}$

$$6 \text{ B. } \mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+16}\right\} = \mathcal{L}^{-1}\left\{\frac{2s}{s^2+16} - \frac{1}{s^2+16}\right\} \\ = 2\cos(4t) - \frac{1}{4}\sin(4t)$$

$$6 \text{ C. } \mathcal{L}^{-1}\left\{\frac{s}{s^2+10s+41}\right\} = \mathcal{L}^{-1}\left\{\frac{s+5-5}{(s+5)^2+16}\right\} = e^{-5t}\cos(4t) - \frac{5}{4}\sin(4t)e^{-5t}$$

$$4 \text{ D. } \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\} = \frac{1}{2}u(t-2)(t-2)^2$$

4. A. Write the function using unit step functions.  $f(t) = \begin{cases} t, & 0 \leq t < 4 \\ 8, & t \geq 4 \end{cases}$

$$f(t) = t - t u(t-4) + 8 u(t-4)$$

B. Find the Laplace transform of the function  $f(t)$ .

$$6 \quad F(s) = \frac{1}{s^2} - e^{-4s} \left[ \frac{1}{s^2} + \frac{4}{s} \right] + \frac{8}{s} e^{-4s}$$

5. Solve the IVP  $y''' + 4y' = e^{2t}$ ,  $y(0) = 1$ ,  $y'(0) = 0$ , and  $y''(0) = 0$  using Laplace transforms.

$$10 \quad S^3 Y - S^2 y^{(0)} - S y^{(1)} - y^{(2)} + 4[sY - y^{(0)}] = \frac{1}{s-2}$$

$$S^3 Y - S^2 + 4sY - 4 = \frac{1}{s-2}$$

$$(S^3 + 4s)Y = \frac{1}{s-2} + S^2 + 4 \quad Y = \frac{\frac{1}{s-2} + S^2 + 4}{S^3 + 4s}$$

$$Y = \frac{1}{16} \frac{S}{S^2+4} - \frac{1}{8} \frac{1}{S^2+4} + \frac{1}{16} \frac{1}{S-2} + \frac{7}{8s} \\ = \frac{1}{16} \cos(2t) - \frac{1}{16} \sin(2t) + \frac{1}{16} e^{2t} + \frac{7}{8}$$

6. Use Laplace transforms to find the charge  $q(t)$  on the capacitor in an RC series circuit when

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$$q(0) = 0, R = 25\Omega, C = 0.01 \text{ f}, \text{ and } E(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 5, & 1 \leq t < 3 \\ 0, & t \geq 3 \end{cases} \quad (\text{Hint: } R \frac{dq}{dt} + \frac{1}{C} q = E(t))$$

$$25 q' + 100 q = 5 u(t-1) - 5 u(t-3)$$

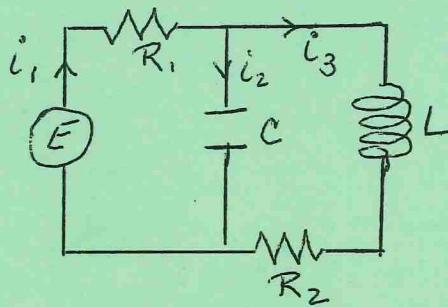
$$25[sQ - q(0)] + 100Q = \frac{5}{s}e^{-s} - \frac{5}{s}e^{-3s}$$

$$[25s+100]Q = \frac{5}{s}e^{-s} - \frac{5}{s}e^{-3s} \quad Q = \frac{\frac{5}{s}e^{-s} - \frac{5}{s}e^{-3s}}{25s+100}$$

$$Q = \frac{1}{20}e^{-s} \frac{1}{s+4} + \frac{1}{20}e^{-s} \frac{1}{s} + \frac{1}{20}e^{-3s} \frac{1}{s+4} - \frac{1}{20}e^{-3s} \frac{1}{s}$$

$$q(t) = \frac{1}{20}e^{4(t-1)} u(t-1) + \frac{1}{20}u(t-1) + \frac{1}{20}e^{4(t-3)} u(t-3) - \frac{1}{20}u(t-3)$$

7. Write a system of differential equations in terms of the charge  $q(t)$ , and the current  $i_3(t)$ , for the electrical circuit shown.



$$E = i_1 R_1 + L i_3' + R_2 i_3$$

$$E = i_1 R_1 + \frac{1}{C} q$$

$$\frac{dq}{dt} = i_2$$

$$\left\{ \begin{array}{l} E = (q' + i_3) R_1 + L i_3' + R_2 i_3 \\ L i_3' + R_2 i_3 - \frac{1}{C} q = 0 \end{array} \right.$$

- 10 8. Solve the system using Laplace transforms.

$$y'' + x + y = 4t$$

$$s^2 Y - s y(0) - y'(0) + X + Y = \frac{4}{s^2}$$

$$x' + y' = e^t$$

$$sX - x(0) + sY - y(0) = \frac{1}{s-1}$$

$$y(0) = 0, y'(0) = 0, x(0) = 3$$

$$s^2 Y + X + Y = \frac{4}{s^2} \Rightarrow X = \frac{4}{s^2} - \frac{1}{s} Y - Y$$

$$sX - 3 + sY = \frac{1}{s-1}$$

$$s \left[ \frac{4}{s^2} - \frac{1}{s} Y - Y \right] - 3 + sY = \frac{1}{s-1}$$

$$x(t) = 4t + e^t - \frac{1}{2}t + t^2 - \frac{2}{3}t^3 - (-e^t - 2 + 4t)$$

$$\frac{4}{s} - s^3 Y - sY - 3 + sY = \frac{1}{s-1}$$

$$-\frac{4}{s^2} - s^3 Y = \frac{1}{s-1} - \frac{4}{s} + 3$$

$$Y = \frac{-1}{s-1} + \frac{1}{s} + \frac{1}{s^2} - \frac{2}{s^3} + \frac{4}{s^4}$$

$$y(t) = -e^t + 1 + t - t^2 + \frac{2}{3}t^3$$

$$x(t) = 2e^t + 1 - t + t^2 - \frac{2}{3}t^3$$

